

WEAKLY CONTRACTIVE MAPS IN ALTERING METRIC SPACES

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ABSTRACT. The weakly contractive metric type fixed point result in Berinde [Nonlin. Anal. Forum, 9 (2004), 45-53] is "almost" covered by the related altering metric one due to Khan et al [Bull. Austral. Math. Soc., 30 (1984), 1-9]. Further extensions of both these results are then provided.

1. INTRODUCTION

Let (X, d) be a complete metric space; and $T \in \mathcal{F}(X)$ be a selfmap of X . [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ stands for the class of all functions from A to B ; when $A = B$, we simply denote $\mathcal{F}(A, A)$ as $\mathcal{F}(A)$]. Put $\text{Fix}(T) = \{z \in X; z = Tz\}$; each element of this set is called *fixed* under T . In the metrical fixed point theory, such points are to be determined by a limit process as follows. Let us say that $x \in X$ is a *Picard point* (modulo (d, T)) when **i)** $(T^n x; n \geq 0)$ is d -convergent, **ii)** $\lim_n (T^n x)$ belongs to $\text{Fix}(T)$. If this happens for each $x \in X$, then T is called a *Picard operator* (modulo d); and, if in addition, **iii)** $\text{Fix}(T)$ is a *singleton* ($z_1, z_2 \in \text{Fix}(T)$ implies $z_1 = z_2$), then T is referred to as a *strong Picard operator* (modulo d); cf. Rus [13, Ch 2, Sect 2.2]. In this perspective, a basic result to the question we deal with is the 1922 one due to Banach [2]: it states that, whenever T is α -contractive (modulo d), i.e.,

$$(a01) \quad d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in X,$$

for some $\alpha \in [0, 1[$, then T is a strong Picard operator (modulo d). This result found a multitude of applications in operator equations theory; so, it was the subject of many extensions. For example, a natural way of doing this is by considering "functional" contractive conditions of the form

$$(a02) \quad d(Tx, Ty) \leq F(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \quad \forall x, y \in X;$$

where $F : R_+^5 \rightarrow R_+$ is an appropriate function. For more details about the possible choices of F we refer to the 1977 paper by Rhoades [12]; see also Turinici [15]. Here, we shall be concerned with a 2004 contribution in the area due to Berinde [4]. Given $\alpha, \lambda \geq 0$, let us say that T is a *weak (α, λ) -contraction* (modulo d) provided

$$(a03) \quad d(Tx, Ty) \leq \alpha d(x, y) + \lambda d(Tx, y), \quad \text{for all } x, y \in X.$$

Theorem 1. *Suppose that T is a weak (α, λ) -contraction (modulo d), where $\alpha \in [0, 1[$. Then, T is a Picard operator (modulo d).*

In a subsequent paper devoted to the same question, Berinde [3] claims that this class of contractions introduced by him is for the first time considered in the

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literature. Unfortunately, his assertion is not true: conclusions of Theorem 1 are "almost" covered by a related 1984 statement due to Khan et al [9], in the context of altering distances. This, among others, motivated us to propose an appropriate extension of the quoted statement; details are given in Section 3. The preliminary material for our device is listed in Section 2. Finally, in Section 4, a "functional" extension of Berinde's result is established. Further aspects will be delineated elsewhere.

2. PRELIMINARIES

Let (X, d) be a metric space. Let us say that the sequence (x_n) in X , *d-converges* to $x \in X$ (and write: $x_n \xrightarrow{d} x$) iff $d(x_n, x) \rightarrow 0$; that is

$$(b01) \quad \forall \varepsilon > 0, \exists p = p(\varepsilon): n \geq p \implies d(x_n, x) \leq \varepsilon.$$

Denote $\lim_n(x_n) = \{x \in X; x_n \xrightarrow{d} x\}$; when this set is nonempty, (x_n) is called *d-convergent*. Note that, in this case, $\lim_n(x_n)$ is a singleton, $\{z\}$; as usually, we write $\lim_n(x_n) = z$. Further, let us say that (x_n) is *d-Cauchy* provided $d(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty, m < n$; that is

$$(b02) \quad \forall \varepsilon > 0, \exists q = q(\varepsilon): q \leq m < n \implies d(x_m, x_n) \leq \varepsilon.$$

Clearly, any *d-convergent* sequence is *d-Cauchy* too; when the reciprocal holds too, (X, d) is called *complete*. Concerning this aspect, note that any *d-Cauchy* sequence $(x_n; n \geq 0)$ is *d-semi-Cauchy*; i.e.,

$$(b03) \quad \rho_n := d(x_n, x_{n+1}) \rightarrow 0 \text{ (hence, } d(x_n, x_{n+i}) \rightarrow 0, \forall i \geq 1), \text{ as } n \rightarrow \infty.$$

The following result about such sequences is useful in the sequel. For each sequence $(z_n; n \geq 0)$ in R and each $z \in R$, put $z_n \downarrow z$ iff $[z_n > z, \forall n]$ and $z_n \rightarrow z$.

Proposition 1. *Suppose that $(x_n; n \geq 0)$ is d-semi-Cauchy, but not d-Cauchy. There exists then $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that*

$$j \leq m(j) < n(j), \quad \alpha(j) := d(x_{m(j)}, x_{n(j)}) > \eta, \quad \forall j \geq 0 \quad (2.1)$$

$$n(j) - m(j) \geq 2, \quad \beta(j) := d(x_{m(j)}, x_{n(j)-1}) \leq \eta, \quad \forall j \geq j(\eta) \quad (2.2)$$

$$\alpha(j) \downarrow \eta \text{ (hence, } \alpha(j) \rightarrow \eta) \text{ as } j \rightarrow \infty \quad (2.3)$$

$$\alpha_{p,q}(j) := d(x_{m(j)+p}, x_{n(j)+q}) \rightarrow \eta, \text{ as } j \rightarrow \infty, \quad \forall p, q \in \{0, 1\}. \quad (2.4)$$

A proof of this may be found in Khan et al [9]. For completeness reasons, we supply an argument which differs, in part, from the original one.

Proof. (Proposition 1) As (b02) does not hold, there exists $\eta > 0$ with

$$A(j) := \{(m, n) \in N \times N; j \leq m < n, d(x_m, x_n) > \eta\} \neq \emptyset, \quad \forall j \geq 0.$$

Having this precise, denote, for each $j \geq 0$,

$$m(j) = \min \text{Dom}(A(j)), \quad n(j) = \min A(m(j)).$$

As a consequence, the couple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ fulfills (2.1). On the other hand, letting the index $j(\eta) \geq 0$ be such that

$$d(x_k, x_{k+1}) < \eta, \quad \forall k \geq j(\eta), \quad (2.5)$$

it is clear that (2.2) holds too. Finally, by the triangular property,

$$\eta < \alpha(j) \leq \beta(j) + \rho_{n(j)-1} \leq \eta + \rho_{n(j)-1}, \quad \forall j \geq j(\eta);$$

and this yields (2.3); hence, the case $(p = 0, q = 0)$ of (2.4). Combining with

$$\alpha(j) - \rho_{n(j)} \leq d(x_{m(j)}, x_{n(j)+1}) \leq \alpha(j) + \rho_{n(j)}, \quad \forall j \geq j(\eta)$$

establishes the case $(p = 0, q = 1)$ of the same. The remaining situations are deductible in a similar way. \square

3. MAIN RESULT

Let X be a nonempty set; and $d(., .)$ be a metric over it [in the usual sense]. Further, let $\varphi \in \mathcal{F}(R_+)$ be an *altering function*; i.e.

(c01) φ is continuous, increasing, and reflexive-sufficient [$\varphi(t) = 0$ iff $t = 0$].

The associated map (from $X \times X$ to R_+)

(c02) $e(x, y) = \varphi(d(x, y)), \quad x, y \in X$

has the immediate properties

$$e(x, y) = e(y, x), \quad \forall x, y \in X \quad (e \text{ is symmetric}) \quad (3.1)$$

$$e(x, y) = 0 \iff x = y \quad (e \text{ is reflexive-sufficient}). \quad (3.2)$$

So, it is a (reflexive sufficient) *symmetric*, under Hicks' terminology [8]. In general, $e(., .)$ is not endowed with the triangular property; but, in compensation to this, one has (as φ is increasing and continuous)

$$e(x, y) > e(u, v) \implies d(x, y) > d(u, v) \quad (3.3)$$

$$x_n \xrightarrow{d} x, \quad y_n \xrightarrow{d} y \quad \text{implies} \quad e(x_n, y_n) \rightarrow e(x, y). \quad (3.4)$$

Suppose in the following that

(c03) (X, d) is complete (each d -Cauchy sequence is d -convergent).

Let $T \in \mathcal{F}(X)$ be a selfmap of X . The formulation of the problem involving $\text{Fix}(T) = \{x \in X; x = Tx\}$ is the already sketched one. In the following, we are trying to solve it in the precise context. Denote, for $x, y \in X$,

$$\begin{aligned} \text{(c04)} \quad M_1(x, y) &= e(x, y), \quad M_2(x, y) = (1/2)[e(x, Tx) + e(y, Ty)], \\ M_3(x, y) &= \min\{e(x, Ty), e(Tx, y)\}, \\ M(x, y) &= \max\{M_1(x, y), M_2(x, y), M_3(x, y)\}. \end{aligned}$$

Further, given $\psi \in \mathcal{F}(R_+)$, we say that T is $(d, e; M, \psi)$ -contractive, provided

$$\text{(c05)} \quad e(Tx, Ty) \leq \psi(d(x, y))M(x, y), \quad \forall x, y \in X, \quad x \neq y.$$

The properties of ψ to be used here write

$$\text{(c06)} \quad \psi \text{ is strictly subunitary on } R_+^0 :=]0, \infty[: \psi(s) < 1, \quad \forall s \in R_+^0$$

$$\text{(c07)} \quad \psi \text{ is right Boyd-Wong on } R_+^0: \limsup_{t \rightarrow s+} \psi(t) < 1, \quad \forall s \in R_+^0.$$

This is related to the developments in Boyd and Wong [6]; we do not give details.

The main result of this exposition is

Theorem 2. *Suppose that T is $(d, e; M, \psi)$ -contractive, where $\psi \in \mathcal{F}(R_+)$ is strictly subunitary and right Boyd-Wong on R_+^0 . Then, T is a strong Picard operator (modulo d).*

Proof. First, let us check the singleton property for $\text{Fix}(T)$. Let $z_1, z_2 \in \text{Fix}(T)$ be such that $z_1 \neq z_2$; hence $\delta := d(z_1, z_2) > 0$, $\varepsilon := e(z_1, z_2) > 0$. By definition,

$$M_1(z_1, z_2) = \varepsilon, \quad M_2(z_2, z_2) = 0, \quad M_3(x, y) = \varepsilon; \quad \text{hence } M(x, y) = \varepsilon.$$

By the contractive condition (written at (z_1, z_2))

$$\varepsilon = e(z_1, z_2) = e(Tz_1, Tz_2) \leq \psi(\delta)M(z_1, z_2) = \psi(\delta)\varepsilon;$$

hence, $1 \leq \psi(\delta) < 1$; contradiction. This established the singleton property. It remains now to verify the Picard property. Fix some $x_0 \in X$; and put $x_n = T^n x_0$, $n \geq 0$. If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, one may assume

$$(c08) \quad \rho_n := d(x_n, x_{n+1}) > 0 \text{ (hence, } \sigma_n := e(x_n, x_{n+1}) > 0), \text{ for all } n.$$

There are several steps to be passed.

I) For the arbitrary fixed $n \geq 0$, we have

$$\begin{aligned} M_1(x_n, x_{n+1}) &= \sigma_n, \\ M_2(x_n, x_{n+1}) &= (1/2)[\sigma_n + \sigma_{n+1}] \leq \max\{\sigma_n, \sigma_{n+1}\}, \\ M_3(x_n, x_{n+1}) &= 0; \text{ hence } M(x_n, x_{n+1}) \leq \max\{\sigma_n, \sigma_{n+1}\}. \end{aligned}$$

By the contractive condition (written at (x_n, x_{n+1})),

$$\sigma_{n+1} \leq \psi(\rho_n) \max\{\sigma_n, \sigma_{n+1}\}, \quad \forall n.$$

This, along with (c08), yields (as ψ is strictly subunitary on R_+^0)

$$\sigma_{n+1}/\sigma_n \leq \psi(\rho_n) < 1, \quad \forall n. \quad (3.5)$$

As a direct consequence,

$$\sigma_n > \sigma_{n+1} \text{ (hence, } \rho_n > \rho_{n+1}), \text{ for all } n.$$

The sequence $(\rho_n; n \geq 0)$ is therefore strictly descending in R_+ ; hence, $\rho := \lim_n(\rho_n)$ exist in R_+ and $\rho_n > \rho, \forall n$. Likewise, the sequence $(\sigma_n = \varphi(\rho_n); n \geq 0)$ is strictly descending in R_+ ; hence, $\sigma := \lim_n(\sigma_n)$ exists; with, in addition, $\sigma = \varphi(\rho)$. We claim that $\rho = 0$. Assume by contradiction that $\rho > 0$; hence $\sigma > 0$. Passing to \limsup as $n \rightarrow \infty$ in (3.5) yields

$$1 \leq \limsup_n \psi(\rho_n) \leq \limsup_{t \rightarrow \rho+} \psi(t) < 1;$$

contradiction. Hence, $\rho = 0$; i.e.,

$$\rho_n := d(x_n, x_{n+1}) \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.6)$$

II) We now show that $(x_n; n \geq 0)$ is d -Cauchy. Suppose that this is not true. By Proposition 1, there exist $\eta > 0$, $j(\eta) \in N$ and a couple of rank sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, in such a way that (2.1)-(2.4) hold. Denote for simplicity $\zeta = \varphi(\eta)$; hence, $\zeta > 0$. By the notations used there, we may write as $j \rightarrow \infty$

$$\lambda_j := e(x_{m(j)+1}, x_{n(j)+1}) = \varphi(\alpha_{1,1}(j)) \rightarrow \zeta.$$

In addition, we have (again under $j \rightarrow \infty$)

$$\begin{aligned} M_1(x_{m(j)}, x_{n(j)}) &= \varphi(\alpha(j)) \rightarrow \zeta \\ M_2(x_{m(j)}, x_{n(j)}) &= (1/2)[\varphi(\rho_{m(j)}) + \varphi(\rho_{n(j)})] \rightarrow 0 \\ M_3(x_{m(j)}, x_{n(j)}) &= \min\{\varphi(\alpha_{0,1}(j)), \varphi(\alpha_{1,0}(j))\} \rightarrow \zeta; \end{aligned}$$

and this, by definition, yields

$$\mu_j := M(x_{m(j)}, x_{n(j)}) \rightarrow \zeta \text{ as } j \rightarrow \infty.$$

From the contractive condition (written at $(x_{m(j)}, x_{n(j)})$)

$$\lambda_j/\mu_j \leq \psi(\alpha(j)), \quad \forall j \geq j(\eta);$$

so that, passing to lim sup as $j \rightarrow \infty$

$$1 \leq \limsup_j \psi(\alpha(j)) \leq \limsup_{t \rightarrow \eta+} \psi(t) < 1;$$

contradiction. Hence, $(x_n; n \geq 0)$ is d -Cauchy, as claimed.

III) As (X, d) is complete, there exists a (uniquely determined) $z \in X$ with $x_n \xrightarrow{d} z$; hence $\gamma_n := d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

Two assumptions are open before us:

i) For each $h \in N$, there exists $k > h$ with $x_k = z$. In this case, there exists a sequence of ranks $(m(i); i \geq 0)$ with $m(i) \rightarrow \infty$ as $i \rightarrow \infty$ such that $x_{m(i)} = z$, $\forall i$; hence, $x_{m(i)+1} = Tz$, $\forall i$. Letting i tends to infinity and using the fact that $(y_i := x_{m(i)+1}; i \geq 0)$ is a subsequence of $(x_i; i \geq 0)$, we get $z = Tz$.

ii) There exists $h \in N$ such that $n \geq h \implies x_n \neq z$. Suppose that $z \neq Tz$; i.e., $\theta := d(z, Tz) > 0$; hence, $\omega := e(z, Tz) > 0$. Note that, in such a case, $\delta_n := d(x_n, Tz) \rightarrow \theta$. From our previous notations, we have (as $n \rightarrow \infty$)

$$\lambda_n := e(x_{n+1}, Tz) = \varphi(\delta_{n+1}) \rightarrow \varphi(\theta) = \omega.$$

In addition (again under $n \rightarrow \infty$),

$$\begin{aligned} M_1(x_n, z) &= \varphi(\gamma_n) \rightarrow 0, \quad M_2(x_n, z) = (1/2)[\sigma_n + \omega] \rightarrow \omega/2 \\ M_3(x_n, z) &= \min\{\varphi(\delta_n), \varphi(\gamma_{n+1})\} \rightarrow 0; \end{aligned}$$

wherefrom,

$$\mu_n := M(x_n, z) \rightarrow \omega/2, \text{ as } n \rightarrow \infty.$$

By the contractive condition (written at (x_n, z))

$$\lambda_n \leq \psi(\gamma_n)\mu_n < \mu_n, \quad \forall n \geq h$$

we then have (passing to limit as $n \rightarrow \infty$), $\omega \leq \omega/2$; hence $\omega = 0$. This yields $\theta = 0$; contradiction. Hence, z is fixed under T and the proof is complete. \square

In particular, the right Boyd-Wong on R_+^0 property of ψ is assured when this function fulfills (c06) and is decreasing on R_+^0 . As a consequence, the following particular version of our main result may be stated.

Theorem 3. *Suppose that T is $(d, e; M, \psi)$ -contractive, where $\psi \in \mathcal{F}(R_+)$ is strictly subunitary and decreasing on R_+^0 . Then, T is a strong Picard operator (modulo d).*

Let $a, b, c \in \mathcal{F}(R_+)$ be a triple of functions. We say that the selfmap T of X is $(d, e; a, b, c)$ -contractive if

$$\begin{aligned} \text{(c09)} \quad e(Tx, Ty) &\leq a(d(x, y))e(x, y) + b(d(x, y))[e(x, Tx) + e(y, Ty)] + \\ &\quad c(d(x, y)) \min\{e(x, Ty), e(Tx, y)\}, \quad \forall x, y \in X, x \neq y. \end{aligned}$$

Denote for simplicity $\psi = a + 2b + c$; it is clear that, under such a condition, T is $(d, e; M; \psi)$ -contractive. Consequently, the following statement is a particular case of Theorem 1 above:

Theorem 4. *Suppose that T is $(d, e; a, b, c)$ -contractive, where the triple of functions $a, b, c \in \mathcal{F}(R_+)$ is such that their associated function $\psi = a + 2b + c$ is strictly subunitary and right Boyd-Wong on R_+^0 . Then, conclusions of Theorem 1 hold.*

In particular, when a, b, c are all decreasing on R_+^0 , the right Boyd-Wong property on R_+^0 holds; note that, in this case, Theorem 4 is also reducible to Theorem 3. This is just the 1984 fixed point result in Khan et al [9].

Finally, it is worth mentioning that the nice contributions of these authors was the starting point for a series of results involving altering contractions, like the one in Dutta and Choudhury [7] or Nashine et al [10]. Some other aspects may be found in Akkouchi [1]; see also Pathak and Shahzad [11].

4. FURTHER ASPECTS

Let again (X, d) be a complete metric space and $T \in \mathcal{F}(X)$ be a selfmap of X . A basic particular case of Theorem 4 corresponds to the choices φ =identity and $[a, b, c$ =constants]. The corresponding form of Theorem 4 is comparable with Theorem 1. However, the inclusion between these is not complete. This raises the question of determining proper extensions of Theorem 1, close enough to Theorem 4. A direct answer to this is provided by

Theorem 5. *Let the numbers $a, b \in R_+$ and the function $K \in \mathcal{F}(R_+)$ be such that*

$$(d01) \quad d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + K(d(Tx, y)), \quad \forall x, y \in X$$

$$(d02) \quad a + 2b < 1 \text{ and } K(t) \rightarrow 0 \text{ as } t \rightarrow 0.$$

Then, T is a Picard map (modulo d).

Proof. Take an arbitrary fixed $u \in X$. By the very contractive condition (written at $(T^n u, T^{n+1} u)$), we have the evaluation

$$d(T^{n+1} u, T^{n+2} u) \leq \lambda d(T^n u, T^{n+1} u), \quad \forall n \geq 0. \quad (4.1)$$

where $\lambda := (a + b)/(1 - b) < 1$. This yields

$$d(T^n u, T^{n+1} u) \leq \lambda^n d(u, Tu), \quad \forall n \geq 0. \quad (4.2)$$

Consequently, $(T^n u; n \geq 0)$ is d -Cauchy; whence (by completeness)

$$T^n u \xrightarrow{d} z := T^\infty u, \text{ for some } z \in X.$$

From the contractive condition (written at $(T^n u, z)$),

$$d(T^{n+1} u, Tz) \leq ad(T^n u, z) + b[d(T^n u, T^{n+1} u) + d(z, Tz)] + K(d(T^{n+1} u, z)), \quad \forall n.$$

Passing to limit as $n \rightarrow \infty$ gives (via (d02)) $d(z, Tz) \leq bd(z, Tz)$; so that, if $z \neq Tz$, one gets $1 \leq b \leq 1/2$, contradiction. Hence $z = Tz$; and the proof is complete. \square

In particular, when $b = 0$ and $K(\cdot)$ is linear ($K(t) = \lambda t$, $t \in R_+$, for some $\lambda \geq 0$), this result is just Theorem 1. Note that, from (4.2), one has for these "limit" fixed points, the error approximation formula (which – under the accepted conditions for our data – is available as well in case of Theorem 4)

$$d(T^n u, T^\infty u) \leq [\lambda^n / (1 - \lambda)] d(u, Tu), \quad \forall n \in N. \quad (4.3)$$

However, the non-singleton property of $\text{Fix}(T)$ makes this "local" evaluation to be without practical effect in Theorem 5, by the highly unstable character of the map $u \mapsto T^\infty u$: even if the distance $d(u, v)$ between two initial approximations would decrease, the distance $d(T^\infty u, T^\infty v)$ between the associated fixed points may not decrease.

Finally, another interesting particular case to consider is that of φ being an arbitrary altering function and $[a, b, c$ =constants]; we do not give details. Further aspects may be found in Bhaumik et al [5] see also Sastry and Babu [14];

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